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THE STABILITY PROBLEM FOR A THEOREM OF ÇRAMER
BY N. A. SAPOGOV

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U. S. DEPARTMENT OF COMMERCE
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The Stability Problem for a Theorem of Cramér

by

N. A. Sapogov

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NBS

The Stability Problem for a Theorem of Cramér

by N. A. Sapogov

(Presented by the Academician, S. N. Bernstein)

Abstract: If the sum of two random variables X_1 and X_2 is almost normally distributed, then each of these random variables is also approximately normally distributed, provided that X_1 and X_2 are either independent or are dependent in a manner specified in this paper. The degree of approximation of the distribution of the summands to the normal distribution is evaluated.

I. Introduction

1. If X_1 and X_2 are two independently and normally distributed random variables, then their sum $X = X_1 + X_2$ is also normally distributed. This is one of the most elementary theorems in the theory of probability. The converse proposition, namely that X_1 and X_2 are also normally distributed if their sum X is normally distributed, is far from being elementary. This proposition was first conjectured by P. Lévy and proved in 1936 by H. Cramér [1]. In addition to the proof given by Cramér, there is an alternate proof due to S. N. Bernstein [2], pp. 427-30. Bernstein uses the more elementary theorem of Liouville instead of Hadamard's theorem on entire functions which Cramér utilized.

However, neither the original proof of Cramér, nor its variant mentioned above, allows us to arrive at any definite conclusion regarding the type of distribution of the random variables X_1 and X_2, if either the distribution of their sum X is not exactly --but only approximately--normal, or if the variables X_1 and X_2 are not entirely independent. The present paper studies these questions. A previous report on the result of this investigation is contained in a note by the author [3].

2. The main result of this investigation may be stated as follows:

THEOREM: Let

$$X = X_1 + X_2 \tag{1.1}$$

be the sum of two independent random variables and assume that the distribution function F(x) of X satisfies the condition

$$\left| F(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt \right| < \varepsilon, \qquad \infty < x < \infty \tag{1.2}$$

where $\varepsilon < 1$ is a given positive number; let also $F_1(x)$ be the distribution function of X_1, and let

$$\int_{-N}^{N} x \, dF_1(x) = a_1,$$

$$\int_{-N}^{N} x^2 \, dF_1(x) - \left(\int_{-N}^{N} x \, dF_1(x)\right)^2 = \sigma_1^2 > 0, \quad N = \sqrt{\ln \frac{1}{\varepsilon}}$$

Then

$$\left| F_1(x) - \frac{1}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}} dx \right| < C\sigma_1^{-\frac{3}{4}} \left(\ln \frac{1}{\varepsilon}\right)^{-\frac{1}{8}}, \quad -\infty < x < \infty \tag{1.3}$$

where C is a constant which does not depend on ε, σ_1 or a_1. An analogous statement can be made regarding $F_2(x)$, the distribution function of X_2.

However, the statement (1.3) is not the ultimate result obtainable and is subject to improvement. At the end of this paper, a generalization of the above result is discussed for the case where X_1 and X_2 are dependent; several other observations are also made.

II. Reduction to Bounded Variables.

3. We shall assume that the median m_1 of X_1 is zero. This is no loss of generality for if $m_1 \neq 0$, then we may investigate $X_1 - m_1$ instead of X_1 and $X_2 + m_1$ instead of X_2. Let now

$$P\{X_1 < 0\} \leq \tfrac{1}{2}, \; P\{X_1 \leq 0\} \geq \tfrac{1}{2} . \tag{2.1}$$

The notation $P\{\;\}$ indicates the probability of the event shown within the curly brackets.

It is easy to show that under these conditions the median m_2 of the value X_2 satisfies the inequality

$$|m_2| < 1, \tag{2.2}$$

for any nonnegative $\varepsilon \leq 1/20$. In fact, from the definition of a median, it follows that

$$P\{X_2 < m_2\} \leq \tfrac{1}{2} , \quad P\{X_2 \leq m_2\} \geq \tfrac{1}{2} .$$

Consequently, taking into consideration also (1.1), (1.2) and (2.1), we have

$$\tfrac{1}{4} \leq P\{X_1 \leq 0;\; X_2 \leq m_2\} \leq P\{X_1 + X_2 \leq m_2\} =$$

$$= P\{X \leq m_2\} \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m_2} e^{-\frac{1}{2}t^2} dt + \varepsilon ,$$

from which it follows that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m_2} e^{-\frac{1}{2}t^2} dt \geq \tfrac{1}{4} - \varepsilon \geq \tfrac{1}{4} - \tfrac{1}{20} = 0.2 .$$

This leads to the inequality

$$m_2 > -1$$

Moreover,

$$P\{X_1 + X_2 < m_2\} \leq P\{X_1 < 0\} + P\{X_2 < m_2\} -$$

$$- P\{X_1 < 0\}.P\{X_2 < m_2\} \leq \frac{3}{4},$$

since

$$u + v - uv \leq \frac{3}{4},$$

if

$$0 \leq u \leq \frac{1}{2} \quad \text{and} \quad 0 \leq v \leq \frac{1}{2}.$$

Consequently

$$\frac{3}{4} \geq P\{X < m_2\} > \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m_2} e^{-\frac{1}{2}t^2} dt - \varepsilon,$$

from which we have

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{m_2} e^{-\frac{1}{2}t^2} dt < \frac{3}{4} + \frac{1}{20} = 0.8.$$

This leads to the inequality

$$m_2 < 1 .$$

4. We shall have to produce a number $a > 0$, such that

$$F_1(a) - F_1(-a) \geq \frac{1}{2} \quad \text{and} \quad F_2(a) - F_2(-a) \geq \frac{1}{2}.$$

We shall show that we may take $a = 3$, if $\varepsilon \leq 1/20$. Let us choose a_2 such that

$$P\{|X_2| < a_2\} \leq \frac{1}{2}, \quad P\{|X_2| \leq a_2\} \geq \frac{1}{2}. \tag{2.3}$$

Then

$$P\{|X_2| \geq a_2\} \geq \frac{1}{2} .$$

Consequently, at least one of the following inequalities is true:

$$P\{X_2 \leq -a_2\} \geq \frac{1}{4} \tag{2.4}$$

or

$$P\{X_2 \geq a_2\} \geq \frac{1}{4} . \tag{2.5}$$

Let us assume the hypothesis (2.4). Taking into consideration the expressions (1.1), (1.2) and (2.1), we have

$$\frac{1}{8} \leq P\{X_1 \leq 0;\ X_2 \leq -a_2\} \leq P\{X \leq -a_2\} \leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a_2} e^{-\frac{1}{2}t^2} dt + \varepsilon .$$

so that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a_2} e^{-\frac{1}{2}t^2} dt \geq \frac{1}{8} - \frac{1}{20} = 0{,}075.$$

Therefore

$$a_2 < 1.5 < 3. \tag{2.6}$$

The hypothesis (2.5) is treated in a similar manner and leads to the same result (2.6).

From (2.3) and (2.6) it follows that

$$F_2(3) - F_2(-3) \geq \frac{1}{2} .$$

A similar inequality is true for $F_1(x)$. For, let us choose a_o satisfying the condition

$$P\{|X_1| < a_o\} \leq \frac{1}{2},\ P\{|X_1| \leq a_o\} \geq \frac{1}{2}. \tag{2.7}$$

Then

$$P\{|X_1| \geq a_o\} \geq \frac{1}{2} ,$$

and two cases arise

$$P\{X_1 \leq -a_o\} \geq \frac{1}{4}$$

or

$$P\{X_1 \geq a_o\} \geq \frac{1}{4}.$$

Both cases are analogous. Let us fix our attention on one of these, say the first case. We have

$$\frac{1}{8} \leq P\{X_1 \leq -a_o;\ X_2 \leq m_2\} \leq P\{X \leq -a_o + m_2\} \leq$$

$$\leq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a_2+m_2} e^{-\frac{1}{2}t^2}\, dt + \varepsilon\,;$$

therefore

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-a_2+m_2} e^{-\frac{1}{2}t^2}\, dt \geq \frac{1}{8} - \frac{1}{20} = 0.075;$$

this leads to the relation

$$a_0 - m_2 < 1.5 \quad .$$

From this inequality and (2.2), we obtain

$$a_0 < 3.$$

Therefore, in view of (2.7)

$$F_1(3) - F_1(-3) \geq \frac{1}{2}\,. \tag{2.8}$$

5. Let us introduce--instead of X_1 and X_2--two new variables X_1^* and X_2^*, such that

$$\begin{aligned} X_i^* &= X_i \quad \text{when } |X_i| \leq N, \\ X_i^* &= 0. \quad \text{when } |X_i| \geq N, \end{aligned} \qquad (i = 1, 2)$$

where $N = \sqrt{\ln \frac{1}{\varepsilon}}$.

Clearly X_1^* and X_2^* are also independent.

We denote by $F_1{}^*(x)$ and $F_2{}^*(x)$ the distribution functions of the variables $X_1{}^*$ and $X_2{}^*$ respectively, and by $F^*(x)$, the distribution function of the sum

$$X^* = X_1^* + X_2^* .$$

We note, first, that

$$|F^*(x) - F(x)| \leq [\int_{|x|>N} dF_1(x) + \int_{|x|>N} dF_2(x)] = \Delta. \tag{2.9}$$

This results from the fact that the probability of the inequality

$$X \neq X^*$$

does not exceed Δ. Let us evaluate Δ. Since

$$P\{X_1 \leq 0;\ X_2 \leq y\} \leq P\{X \leq y\} < \varepsilon + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}t^2}\, dt,$$

then

$$F_2(-N) < 2\varepsilon + \frac{2}{\sqrt{2\pi}} \int_{N}^{\infty} e^{-\frac{1}{2}t^2}\, dt.$$

Similarly, from

$$P\{X_1 \geq 0; X_2 > y\} \leq P\{X > y\} < \varepsilon + \frac{1}{\sqrt{2\pi}} \int_{y}^{\infty} e^{-\frac{1}{2}t^2}\, dt$$

we obtain

$$1 - F_2(N) < 2\varepsilon + \frac{2}{\sqrt{2\pi}} \int_{N}^{\infty} e^{-\frac{1}{2}t^2}\, dt.$$

Therefore

$$\int_{|y|>N} dF_2(y) < 4\varepsilon + \frac{4}{\sqrt{2\pi}} \int_{N}^{\infty} e^{-\frac{1}{2}t^2}\, dt. \tag{2.10}$$

In the very same manner,--remembering (2.2)--, we find that

$$\int_{|y|>N} dF_1(y) < 4\varepsilon + \frac{4}{\sqrt{2\pi}} \int_{N-1}^{\infty} e^{-\frac{1}{2}t^2}\, dt. \tag{2.11}$$

The inequalities (1.2), (2.9), (2.10) and (2.11) lead to

the relation

$$|F^*(x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}t^2} dt| < 9\varepsilon + \frac{8}{\sqrt{2\pi}} \int_{N-1}^{\infty} e^{-\frac{1}{2}u^2} du = \varepsilon_1. \qquad (2.12)$$

III. Investigation of Characteristic Functions

6. Let $f^*(z)$ be the characteristic function of the variable X^*:

$$f^*(z) = E(e^{izX^*}) = \int_{-\infty}^{\infty} e^{izx}\, dF^*(x).$$

Since $|X^*| \leq 2N$, then $f^*(z)$ is an entire function of the complex argument z. Similarly, the characteristic functions

$$f_1^*(z) = E(e^{izX_1^*}) \quad \text{and} \quad f_2^*(z) = E(e^{izX_2^*}) \qquad (3.1)$$

are also entire functions.

Our immediate goal is to find a lower bound for the modulus $|f^*(z)|$, when z is in the circle

$$|z| \leq T = \frac{N}{8} = \frac{1}{8}\sqrt{\ln \frac{1}{\varepsilon}} .$$

It is well known that

$$e^{-\frac{1}{2}z^2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{izx}\, e^{-\frac{1}{2}x^2}\, dx.$$

so that $e^{-\frac{1}{2}z^2}$ is the characteristic function of the normal distribution

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du = \Phi(x).$$

We have therefore

$$|f^*(z) - e^{-\frac{1}{2}z^2}| \leq \int_{-2N}^{2N} e^{izx}\, d[F^*(x) - \Phi(x)] +$$

$$+ \frac{1}{\sqrt{2\pi}} \left| \int_{|x| \geqslant 2N} e^{izx} e^{-\frac{1}{2}x^2} dx \right|. \tag{3.2}$$

The first term of the right-hand side is now evaluated:

$$\left| \int_{-2N}^{2N} e^{izx}\, d[F^*(x) - \Phi(x)] \right| =$$

$$= \left| \{e^{izx}[F^*(x) - \Phi(x)]\}_{-2N}^{2N} - \int_{-2N}^{2N} [F^*(x) - \Phi(x)] iz e^{izx} dx \right| \leq$$

$$\leq 2e^{N^2/4}\varepsilon_1 + \frac{N^2}{2} e^{N^2/4}\varepsilon_1 = 2\varepsilon_1 e^{N^2/4}(1 + \frac{N^2}{4}), \tag{3.3}$$

In (3.3) it is assumed that $|z| \leqslant T = N/8$ and ε_1 is defined by (2.12). The second term of the right-hand side of inequality (3.2) is evaluated in the following manner:

$$\frac{1}{\sqrt{2\pi}} \left| \int_{|x| \geqslant 2N} e^{izx} e^{-\frac{1}{2}x^2} dx \right| < \frac{2}{\sqrt{2\pi}} \int_{2N}^{\infty} e^{Tx - \frac{1}{2}x^2} dx <$$

$$< \sqrt{\frac{2}{\pi}} \frac{e^{\frac{1}{2}T^2}}{2N - T} \int_{2N-T}^{\infty} u e^{-u^2/2} du < \frac{1}{2N} e^{-7N^2/4} . \tag{3.4}$$

Putting together the inequalities (3.2), (3.3) and (3.4), we obtain

$$|f^*(z) - e^{-\frac{1}{2}z^2}| < 2\varepsilon_1 e^{N^2/4}(1 + \frac{N^2}{4}) + \frac{1}{2N} e^{-7N^2/4} <$$

$$< \left(18\varepsilon + \frac{16}{\sqrt{2\pi}(N-1)} e^{-\frac{1}{2}(N-1)^2}\right) e^{N^2/4}(1 + \frac{N^2}{4}) + \frac{1}{2N} e^{-7N^2/4} <$$

$$< 18\varepsilon N^2 e^{N^2/4} + N^2 e^{-21N^2/128} + e^{-7N^2/4} =$$

$$= 18\varepsilon^{3/4} \ln \frac{1}{\varepsilon} + \varepsilon^{21/128} \ln \frac{1}{\varepsilon} + \varepsilon^{7/4} \tag{3.5}$$

provided $N = \sqrt{\ln \frac{1}{\varepsilon}}$ is sufficiently large and $|z| \leq T = \frac{N}{8}$.
In the circle under consideration the following relations hold

$$|e^{-\frac{1}{2}z^2}| \geq e^{-N^2/128} = \varepsilon^{1/128} \quad . \tag{3.6}$$

From (3.5) and (3.6) we obtain

$$|f^*(z) - e^{-z^2/2}| < \tfrac{1}{2}|e^{-\frac{1}{2}z^2}| \tag{3.7}$$

Equation(3.7) is true when $|z| \leq T$, and if ε is so small that

$$18\varepsilon^{3/4} \ln \frac{1}{\varepsilon} + \varepsilon^{21/128} \ln \frac{1}{\varepsilon} + \varepsilon^{7/4} < \tfrac{1}{2}\varepsilon^{1/128} \quad .$$

Therefore in the same circle $|z| \leq T$

$$|f^*(z)| > \frac{1}{2} e^{-\frac{1}{2}|z^2|} \tag{3.8}$$

From this we conclude that $f^*(z)$ has no zeros in the circle $|z| \leq T$.

7. According to (3.1) we have

$$f^*(z) = f_1^*(z)\, f_2^*(z) \;.$$

Therefore both functions $f_1^*(z)$ and $f_2^*(z)$ have no zeros in the circle $|z| \leq T$, so that their logarithms

$$\phi_1(z) = \ln f_1^*(z) \quad \text{and} \quad \phi_2(z) = \ln f_2^*(z)$$

are regular functions.

From (3.7) it follows that

$$|f^*(z)| = |f_1^*(z)\ f_2^*(z)| < \frac{3}{2}\, e^{\frac{1}{2}|z|^2} \quad ,|z| \leq T. \tag{3.9}$$

Let $z = t + is$, where t and s are real numbers. Then, in view of (2.8), we have

$$F_1^*(3) - F_1^*(-3) \geq \frac{1}{2} \text{ and therefore}$$

$$f_1^*(is) = \int_{-\infty}^{\infty} e^{-sx}\, dF_1^*(x) \geq \int_{-3}^{3} e^{-sx} dF_1^*(x) \geq \tfrac{1}{2}\, e^{-3|s|}, \tag{3.10}$$

From (3.9) and (3.10) it follows that

$$|f_2^*(z)| \leq f_2^*(is) = \frac{f^*(is)}{f_1^*(is)} \leq 3e^{3|z| + \frac{1}{2}|z^2|}. \tag{3.11}$$

Similarly, we find that

$$|f_1^*(z)| \leq 3\, e^{3|z| + \frac{1}{2}|z^2|}, \quad |z| \leq T. \tag{3.12}$$

Moreover, the inequalities (3.8), (3.11), and (3.12) allow us to evaluate the moduli $|f^*_1(z)|$ and $|f^*_2(z)|$. In fact,

$$\frac{1}{2}\, e^{-\frac{1}{2}|z^2|} < |f^*(z)| = |f_1^*(z)\ f_2^*(z)|;$$

therefore, and because of (3.11)

$$|f_1^*(z)| > \frac{\frac{1}{2} e^{-\frac{1}{2}|z^2|}}{|f_2^*(z)|} \geq \frac{1}{6}\, e^{-3|z| - |z|^2} \tag{3.13}$$

Similarly, by utilizing (3.12), we obtain

$$|f_2^*(z)| > \frac{1}{6}\, e^{-3|z| - |z|^2}, \quad |z| \leq T. \tag{3.14}$$

8. Let us note first, that

$$\frac{1}{\sigma_1^2} \leq \frac{1}{4} \left(\ln \frac{1}{\varepsilon}\right)^{1/3}. \tag{3.15}$$

We see then from the inequalities (3.11), (3.12), (3.13) and (3.14) that for large values of T, the logarithms of $f_1^*(z)$ and $f_2^*(z)$ do not differ much from certain polynomials of the second degree, as long as $|z| \leq T_1 = \sqrt[4]{\frac{T}{\sigma_1}}$.

Let us use the well-known formula representing a function which is regular within a circle, by means of its real part given on the circumference of the circle (Schwartz's formula)

$$f(z) = iv(0) + \frac{1}{2\pi} \int_0^{2\pi} u(R,\phi) \frac{\xi + z}{\xi - z} \, d\phi ,$$

where $f(z) = u(r,\psi) + iv(r,\psi)$ is a function regular in the circle $|z| = |re^{i\psi}| \leq R$, and $\xi = Re^{i\phi}$. Applying this to the function $\phi_1(z) = \ln f_1^*(z)$ and assuming R = T, we have

$$\phi_1(z) = i \operatorname{Im} \phi_1(0) + \frac{1}{2\pi} \int_0^{2\pi} \ln|f_1^*(\xi)| \frac{\xi + z}{\xi - z} d\phi,$$

whence

$$\phi_1'''(z) = \frac{6}{\pi} \int_0^{2\pi} \ln|f_1^*(\xi)| \frac{\xi \, d\phi}{(\xi - z)^4} \quad . \tag{3.16}$$

From the inequalities (3.12) and (3.13) it follows that

$$\left| \ln |f_1^*(\xi)| \right| < (|\xi| + 3)^2, \tag{3.17}$$

for any

$$|\xi| \leq T.$$

Let us recall that we are considering only those values of z which according to (3.15), satisfy the condition

$$|z| \leq T_1 = \sqrt[4]{\frac{T}{\sigma_1}} \leq 4T \left(\ln \frac{1}{\varepsilon}\right)^{-1/3} \tag{3.18}$$

Keeping in mind (3.17), we have from (3.16)

$$|\phi_1'''(z)| < \frac{12T(T+3)^2}{(T-|z|)^4} .$$

Whence, in view of (3.18), we have for small values of $\varepsilon > 0$

$$|\phi_1'''(z)| < \frac{c_1 T^3}{T^4} = \frac{c_1}{T}$$

($c_1, c_2, \ldots$ are constants).

After three successive integrations of this inequality, we obtain

$$|\phi_1(z) - \alpha_1 - i\beta_1 z + \tfrac{1}{2}\gamma_1 z^2| < \frac{c_1 T_1^3}{T} , \qquad (3.19)$$

where α_1, β_1, γ_1 are certain constants, $|z| \le T_1$.

A similar inequality may be derived for $\phi_2(z)$, the logarithm of the characteristic function $f_2^*(z)$ of the variable X_2^* .

IV. Proof of the Basic Theorem

9. Since $\phi_1(0) = 0$, it follows from (3.19) that

$$|\alpha_1| < \frac{c_1 T_1^3}{T}$$

therefore

$$|\phi_1(z) - i\beta_1 z + \tfrac{1}{2}\gamma_1 z^2| < \frac{2c_1 T_1^3}{T} .$$

Consequently,

$$f_1^*(z) = e^{\phi_1(z)} = e^{i\beta_1 z - \frac{1}{2}\gamma_1 z^2 + H(z)},$$

where $H(z)$ is regular for $|z| < T_1$ and in this circle

$$|H(z)| < 2c_1 \frac{T_1^3}{T} \le 2c_1$$

since $T_1 \leq \sqrt[3]{T}$. But in this case

$$e^{H(z)} = 1 + H_1(z),$$

where

$$H_1(z) = \lambda_1 z + \lambda_2 z^2 + \dots \tag{4.1}$$

is a certain entire function, such that in the circle $|z| \leq T_1$

$$|H_1(z)| < c_2 \frac{T_1^{\,3}}{T} \quad . \tag{4.2}$$

Therefore

$$f_1^*(z) = e^{i\beta_1 z - \frac{1}{2}\gamma_1 z^2} (1 + H_1(z)) = [1 + (i\beta_1 z - \frac{1}{2}\gamma_1 z^2) + \\ + \frac{1}{2!}(i\beta_1 z - \frac{1}{2}\gamma_1 z^2)^2 + \dots](1 + \lambda_1 z + \lambda_2 z^2 + \dots). \tag{4.3}$$

Here

$$\lambda_1 = \frac{1}{2\pi i} \int_{|z|=T_1} \frac{H_1(z)dz}{z^2}, \qquad \lambda_2 = \frac{1}{2\pi i} \int_{|z|=T_1} \frac{H_1(z)dz}{z^3}.$$

Consequently, considering (4.2), we find

$$|\lambda_1| < c_2 \frac{T_1}{T}, \qquad |\lambda_2| < \frac{c_2}{T} .$$

But as an entire function $f_1^*(z)$ allows an expansion

$$f_1^*(z) = 1 + ia_1 z - \frac{a_2}{2!} z^2 + \dots + \frac{i^k a_k}{k!} z^k + \dots,$$

where

$$a_k = M[(X_1^*)^k], \quad k = 1, 2, \dots$$

Comparing this expansion, with (4.3), we obtain

$$\beta_1 - i\lambda_1 = a_1; \quad \gamma_1 + \beta_1^2 - 2i\beta_1\lambda_1 - 2\lambda_2 = a_2$$

Remembering the previously obtained values of λ_1 and λ_2, we have

$$|\beta_1 - a_1| = |\lambda_1| < c_2 \frac{T_1}{T}$$

and for small $\varepsilon > 0$

$$|\gamma_1 - \sigma^2| = |2\lambda_2 - \lambda_1^2| < c_3$$

since $T_1^2/T \to 0$, as $\varepsilon \to 0$. Here $\sigma_1^2 = a_2 - a_1^2$ is the variance of X_1^*. From the above results, it follows that

$$|\phi_1(z) - ia_1 z + \tfrac{1}{2}\sigma_1^2 z^2| < 2c_1 \frac{T_1^3}{T} + c_4 \frac{T_1^2}{T} < c_5 \frac{T_1^3}{T} . \qquad (4.4)$$

A similar inequality is true for $\phi_2(z) = \ln f_2^*(z)$. Denote by

$$g_1(z) = e^{ia_1 z - \frac{1}{2}\sigma_1^2 z^2} .$$

Then we conclude from (4.4) that

$$f_1^*(z) = g_1(z)\big(1 + H_2(z)\big), \qquad (4.5)$$

where $H_2(z)$ is an entire function, such that in the circle $|z| \leq T_1$

$$|H_2(z)| < c_6 \frac{T_1^3}{T} . \qquad (4.6)$$

Moreover, $H_2(0) = 0$, since $f_1^*(0) = 1$.

10. The relation (4.5), which indicates the degree of closeness between the characteristic functions $f_1^*(z)$ and $g_1(z)$, also allows us to evaluate the deviation of the corresponding distributions from each other. For this purpose, we shall make use of the Theorem of Esseen [4]. See also [5] pp. 212-214.

THEOREM (Esseen). Let A, L, and λ be positive constants. Let F(x) be a non-decreasing function and G(x) a function of bounded variation. If:

1) $F(-\infty) = G(-\infty)$, $F(\infty) = G(\infty)$;

2) $\int_{-\infty}^{\infty} |F(x) - G(x)|\, dx < \infty$;

3) The derivative $G'(x)$ exists for all x and $|G'(x)| \leq A$;

4) $\int_{-L}^{L} \left| \frac{f(t) - g(t)}{t} \right| dt = \lambda$,

where $f(t)$ and $g(t)$ are the characteristic functions of $F(x)$ and $G(x)$ respectively, then

$$|F(x) - G(x)| \leq k \frac{\lambda}{2\pi} + c(k) \frac{A}{L},$$

for any $k > 1$, where $c(k)$ is a finite positive number, defined by k.

This theorem is immediately applicable to our case, if we assume

$$L = T_1,\ f(t) = f_1^*(t),\ g(t) = g_1(t).$$

In fact, from (4.5), we have

$$\left| \frac{f_1^*(t) - g_1(t)}{t} \right| \leq \left| \frac{H_2(t)}{t} \right|,$$

for any real t. Since $H_2(0) = 0$, then $H_2(z)/z$ is an entire function and, in view of (4.6), the following is true for $|z| = T_1$

$$\left| \frac{H_2(z)}{z} \right| \leq c_6 \frac{T_1^2}{T}.$$

Since the modulus of a regular function reaches its maximum on the boundary of the region, then for $|z| \leq T_1$, we also have

$$\left| \frac{H_2(z)}{z} \right| \leq c_6 \frac{T_1^2}{T}.$$

Therefore

$$\int_{-T_1}^{T_1} \left| \frac{f_1^*(t) - g_1(t)}{t} \right| dt \leq \int_{-T_2}^{T_2} \left| \frac{H_2(t)}{t} \right| dt \leq 2c_6 \frac{T_1^2}{T} = \lambda .$$

Moreover, the distribution corresponding to the characteristic function $g_1(t)$, namely

$$\frac{1}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}} dx = G(x)$$

has a bounded derivative

$$|G'(x)| = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}} \leq \frac{1}{\sigma_1\sqrt{2\pi}} .$$

In consequence, the Theorem of Esseen allows us to state that

$$\left| F_1^*(x) - \frac{1}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}} dx \right| < c_7 \frac{T_1^3}{T} + \frac{c_8}{T_1\sigma_1} =$$

$$= \frac{c_9}{\sigma_1^{\frac{3}{4}} (\ln \frac{1}{\varepsilon})^{\frac{1}{8}}} , \quad -\infty < x < \infty . \tag{4.7}$$

In the above, we assumed the hypothesis (3.15). If that does not hold, in other words, if

$$\frac{1}{\sigma_1^2} > \frac{1}{4}(\ln \frac{1}{\varepsilon})^{\frac{1}{3}} ,$$

then the inequality (4.7) is trivial, provided the constant c_9 is larger than $2^{3/4}$, for under those conditions

$$\frac{c_9}{\sigma_1^{\frac{3}{4}} (\ln \frac{1}{\varepsilon})^{\frac{1}{8}}} > 1 .$$

In order to obtain a proof of (1.3), it is sufficient to note that

$$\left| F_1(x) - F_1^*(x) \right| \leq \int_{|x| \geq \sqrt{\ln \frac{1}{\varepsilon}}} dF_1(x) < 4\varepsilon + \frac{4}{\sqrt{2\pi}} \int_{\sqrt{\ln \frac{1}{\varepsilon}} - 1}^{\infty} e^{-\frac{1}{2}u^2}\, du.$$

The often repeated condition that ε must be sufficiently small does not have any bearing in the final formulation of the theorem, for we can obviate this condition, by raising the value of the constant C.

Let us note, in particular, that if $\sigma_1 \geq \sigma > 0$ for all sufficiently small $\varepsilon > 0$, where σ is a constant, then we have

$$\left| F_1(x) - \frac{1}{\sigma_1\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-a_1)^2}{2\sigma_1^2}}\, dx \right| < \frac{1}{(\ln \frac{1}{\varepsilon})^{\omega}}$$

for any $\omega < 1/8$, provided $\varepsilon > 0$ is sufficiently small.

5. Supplementary Notes.

11. Let us turn our attention to the fact that the theorem just proven may be paraphrased in terms of moments. In fact, the closeness of the distribution function F(x) of the quantity X to the normal distribution function indicates that several first moments of F(x) and the corresponding moments of the normal distribution are also close. It follows then that several first moments of the distribution $F_1(x)$ and the corresponding first moments of some normal distribution $\Phi_1(x)$ are also close.

In our investigation we found that the hypothesis assuming complete independence of X_1 and X_2 was not essential.

Let now X be the sum of two dependent variables X_1 and X_2, whose distribution functions are a priori $F_1(x)$ and $F_2(x)$. Assume that the distribution function $F(x)$ satisfies the condition

$$|F(x) - \Phi(x)| < \varepsilon', \quad -\infty < x < \infty ,$$

and that the dependence between X_1 and X_2 is such that

$$\left| P\{X_1 + X_2 < x\} - \int_{-\infty}^{\infty} F_1(x-y)dF_2(y)\right| < \varepsilon'', \quad -\infty < x < \infty .$$

We consider two independent variables X_1 and X_2 with distribution functions $F_1(x)$ and $F_2(x)$, and assume that

$$\bar{X}_1 + \bar{X}_2 = \bar{X} ,$$

We then have

$$|F(x) - \Phi(x)| < \varepsilon' + \varepsilon'',$$

where $F(x)$ is the distribution function of X. Now we may apply the Theorem of Section II. A corollary of Cramér's theorem:

Let the sum X of two random variables X_1 and X_2 be subject to the Gaussian law but the two addends may, in general, be dependent. If there exists a constant a, such that x_1 and x_2-aX_1 are independent, then X_1 and X_2 are normally correlated.

Indeed, in this case the quantities $(1+a)X_1$ and X_2-aX_1 are independent, and moreover, their sum is equal to X and is normally distributed. Then each of the above quantities is individually subject to the Gaussian law, and it

follows that X_1 and X_2 are connected by a normal correlation.

Finally, it might be of value to note a fact which, though it is not in the direct line of our investigation, is closely connected with Cramér's theorem. Similar to the Gaussian distribution, Poisson's distribution possesses a property, stated in a theorem of D. A. Raikov [6], which is completely analogous to that of Cramér. At the same time, both of these distributions are the limits of binomial distributions. For these, a theorem holds which is analogous to Cramer's and Raikov's. That is: if the sum of two independent random variables is binomially distributed, then each addend is also binomially distributed (or is non-random). This results from the fact that the generating function $(p+tq)^n$ of the binomial distribution has polynomial divisor's of the same type.

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Corrigenda

to N.B.S. Report 2230

The Stability Problem for a theorem of [illegible]

Page	Line	Now reads	Should read
2	16	$\infty < x < \infty$	$-\infty < x < \infty$
5	5	$\int_{-\infty}^{-a_2+m_2}$	$\int_{-\infty}^{-a_0+m_2}$
5	10	$\int_{-\infty}^{-a_2+m_1}$	$\int_{-\infty}^{-a_0+m_2}$
6	20	$\lvert X\rvert \geq N$	$\lvert X_1\rvert > N$
9	1	$\leq \int_{-2N}^{2N} e^{itx} d[F^*(x) - \Phi(x)]$	$\leq \left\lvert \int_{-2N}^{2N} e^{itx} d[F^*(x) - \Phi(x)] \right\rvert$
11	15	$= \left\lvert f_1(z) \; f_2^*(z) \right\rvert$;	$= \left\lvert f_1^*(z) \; f_2^*(z) \right\rvert$
11	18	let us note first	let us assume first
17	1	$\leq \int_{-T_2}^{T_2} \left\lvert \frac{H_2(t)}{t} \right\rvert dt \leq 2c_6 \frac{T_1^2}{T} = \lambda$	$\leq \int_{-T_1}^{T_1} \left\lvert \frac{H_2(t)}{t} \right\rvert dt \leq 2c_6 \frac{T_1^2}{T} = \lambda$

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